Appendix A

Integral $\int_0^t e^{-\sigma t} \sin \omega t dt$

The differential of a product of two functions u and v is $d(uv) = udv + vdu$. When integrated it becomes $uv = \int udv + \int vdu$. By rearranging we get

$$\int udv = uv - \int vdu \tag{A.1}$$

Lets substitute as follows.

$$\begin{array}{r|l} u = e^{-\sigma t} & dv = \sin \omega t dt \\ du = -\sigma e^{-\sigma t} dt & v = -\frac{1}{\omega} \cos \omega t \end{array}$$

Then, (A.1) can be written as follows

$$\int_0^t e^{-\sigma t} \sin \omega t dt = -\frac{1}{\omega} e^{-\sigma t} \cos \omega t|_0^t - \frac{\sigma}{\omega} \int_0^t e^{-\sigma t} \cos \omega t dt \tag{A.2}$$

Here, we notice that the second term on the RHS can be evaluated according to the same procedure by substituting as follows.

$$\begin{array}{r|c} u = e^{-\sigma t} & dv = \cos \omega t dt \\ du = -\sigma e^{-\sigma t} dt & v = \frac{1}{\omega} \sin \omega t \end{array}$$

Therefore, we can write (A.2) with the second term on the RHS being integrated by parts as follows.

$$\begin{aligned} \int_0^t e^{-\sigma t} \sin \omega t dt &= -\left[\frac{1}{\omega} e^{-\sigma t} \cos \omega t - \frac{1}{\omega}\right] \\ & \quad -\frac{\sigma}{\omega}\left[\frac{1}{\omega} e^{-\sigma t} \sin \omega t|_0^t - \int_0^t \frac{-\sigma}{\omega} e^{-\sigma t} \sin \omega t dt\right] \\ &= \frac{1}{\omega} - \frac{1}{\omega} e^{-\sigma t} \cos \omega t - \frac{\sigma}{\omega^2} e^{-\sigma t} \sin \omega t - \frac{\sigma^2}{\omega^2} \int_0^t e^{-\sigma t} \sin \omega t dt \end{aligned}$$

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\begin{aligned}
(\omega^2 + \sigma^2) \int_0^t e^{-\sigma t} \sin \omega t dt &= \omega - \omega e^{-\sigma t} \cos \omega t - \sigma e^{-\sigma t} \sin \omega t \\
&= \frac{\omega}{\omega^2 + \sigma^2} \\
&\quad - \frac{e^{-\sigma t}}{\sqrt{\sigma^2 + \omega^2}} \left(\frac{\omega}{\sqrt{\sigma^2 + \omega^2}} \cos \omega t + \frac{\sigma}{\sqrt{\sigma^2 + \omega^2}} \sin \omega t \right) \\
&= \frac{\omega}{\omega^2 + \sigma^2} - \frac{e^{-\sigma t}}{\sqrt{\sigma^2 + \omega^2}} \sin(\omega t + \phi) \qquad \text{(A.3)}
\end{aligned}
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where $\tan \phi = \frac{\omega}{\sigma}$